Generalizations of The Chung-Feller Theorem II

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Abstract

The classical Chung-Feller theorem [2] tells us that the number of Dyck paths of length n with m flaws is the n-th Catalan number and independent on m. L. Shapiro [9] found the Chung-Feller properties for the Motzkin paths. Mohanty's book [5] devotes an entire section to exploring Chung-Feller theorem. Many Chung-Feller theorems are consequences of the results in [5]. In this paper, we consider the (n, m)-lattice paths. We study two parameters for an (n, m)-lattice path: the non-positive length and the rightmost minimum length. We obtain the Chung-Feller theorems of the (n, m)-lattice path on these two parameters by bijection methods. We are more interested in the pointed (n, m)-lattice paths. We investigate two parameters for an pointed (n, m)-lattice path: the pointed non-positive length and the pointed rightmost minimum length. We generalize the results in [5]. Using the main results in this paper, we may find the Chung-Feller theorems of many different lattice paths.

Keywords: Chung-Feller Theorem; Dyck path; Motzkin path

1 Introduction

Let $\mathbb{Z}$ denote the set of the integers and $[n] := \{1, 2, \ldots, n\}$. We consider n-Dyck paths in the plane $\mathbb{Z} \times \mathbb{Z}$ using *up* $(1, 1)$ and *down* $(1, -1)$ steps that go from the origin to the point (2n,0). We say n the *semilength* because there are $2n$ steps. An *n-flawed* path is an n-Dyck path that contains some steps under the x-axis. The number of n-Dyck path that never pass below the x-axis is the n-th Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$. Such paths are called the *Catalan paths of length n*. A Dyck path is called a (n, r)-*flawed* path if it contains r up steps under the x-axis and its semilength is n. Clearly, $0 \leq r \leq n$. The classical Chung-Feller theorem [2] says that the number of the (n, r)-flawed paths is equal to c_n and independent on r.

The classical Chung-Feller Theorem were proved by MacMahon [7]. Chung and Feller reproved this theorem by using analytic method in [2]. T.V.Narayana [8] showed the Chung-Feller Theorem

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by combinatorial methods. S. P. Eu et al. [3] proved the Chung-Feller Theorem by using the Taylor expansions of generating functions and gave a refinement of this theorem. In [4], they gave a strengthening of the Chung-Feller Theorem and a weighted version for Schröder paths. Y.M. Chen [1] revisited the Chung-Feller Theorem by establishing a bijection.

Mohanty's book [5] devotes an entire section to exploring Chung-Feller theorem. We state the result from [5] as the following lemma.

Lemma 1.1 *[5] Given a positive integer n, let $Y = (y_1, \ldots, y_{n+1})$ be a sequence of integers with $1-n \le y_i \le 1$ for all $i \in [n+1]$ such that $\sum\limits_{i=1}^{n+1} y_i = 1$. Furthermore, let $E(Y) = |\{i \mid \sum\limits_{j=1}^{i} y_j \le 0\}|$. Let Y_i be the i-th cyclic permutation of Y (i.e., $Y_i = (y_i, y_{i+1}, \ldots, y_{n+i+1})$ with $y_{n+r+1} = y_r$). Then there exists a permutation $i_1, \ldots, i_{n+1}$ on the set $[n+1]$ such that $E(Y_{i_1}) > E(Y_{i_2}) > \cdots > E(Y_{i_{n+1}})$.*

Many Chung-Feller theorems are consequences of lemma 1.1. First, let ϕ be a mapping from $\mathbb{Z}$ to $\mathbb{P}$, where $\mathbb{P}$ is a set of all the positive integers. Let the sequence $Y = (y_1, \ldots, y_{n+1})$ satisfy the conditions in Lemma 1.1. Using $(\phi(y_i), y_i)$ steps, we can obtain a lattice path $P(Y) = (\phi(y_1), y_1)(\phi(y_2), y_2)\ldots(\phi(y_{n+1}), y_{n+1})$ in the plane $\mathbb{Z} \times \mathbb{Z}$ that go from the origin to the point $(\sum\limits_{i=1}^{n+1} \phi(y_i), 1)$. Using Lemma 1.1, we will derive the classical Chung-Feller theorem for Dyck paths if we let $y_i \in \{1, -1\}$ and set $\phi(y) = 1$ for all $y \in \mathbb{Z}$; we will derive the Chung-Feller theorem for Schröder paths if we let $y_i \in \{1, 0, -1\}$ and set $\phi(0) = 2$ and $\phi(y) = 1$ for $y \neq 0$; we will derive the Chung-Feller theorem for Motzkin paths if we let $y_i \in \{1, 0, -1\}$ and set $\phi(0) = 1$ and $\phi(y) = 1$ for $y \neq 0$ and so on.

How to derive the Chung-Feller theorem for lattice paths in the plane $\mathbb{Z} \times \mathbb{Z}$ using $(1, -1)$, $(1, 1)$, $(1, 0)$, $(2, 0)$ steps? For answering this problem, the authors of this paper [6] proved the Chung-Feller theorems for three classes of lattice paths by using the method of the generating functions. It is interesting that these Chung-Feller theorems can't be derivable as a special case from lemma 1.1. This implies that we may generalize the results of Lemma 1.1.

In this paper, first we give the definition of the (n, m)-lattice paths. We consider two parameters for an (n, m)-lattice path: the non-positive length and the rightmost minimum length. Using bijection methods, we obtain the Chung-Feller theorems of the (n, m)-lattice path on these two parameters. Then we study the pointed (n, m)-lattice paths. We investigate two parameters for an pointed (n, m)-lattice path: the pointed non-positive length and the pointed rightmost minimum length. We give generalizations of the results in [5] and prove the Chung-Feller theorems of the pointed (n, m)-lattice path on these two parameters. Finally, using the main theorems of this paper, we may find the Chung-Feller theorems of many different (n, m)-lattice paths.

This paper is organized as follows. In Section 2, we focus on the (n, m)-lattice paths. Using bijection methods, we obtain the Chung-Feller theorems of the (n, m)-lattice path. In Section 3,

we study the pointed (n,m)-lattice paths and give generalizations of the results in [5]. In Section 4, using the main theorems of this paper, we find the Chung-Feller theorems of many different (n,m)-lattice paths.

2 The (n,m)-lattice paths

Throughout the paper, we always let n and m be two positive integers with $m \geq n+1$. In this section, we will consider the (n,m)-lattice paths. We will define two parameters for an (n,m)-lattice path: the non-positive length and the rightmost minimum length. Using bijection methods, we will obtain the Chung-Feller theorems of the (n,m)-lattice path on these two parameters. First, we give the definition of the (n,m)-lattice paths as follows.

Definition 2.1 *An (n,m)-lattice paths P is a sequence of the vectors $(x_1,y_1)(x_2,y_2)\dots(x_{n+1},y_{n+1})$ in $\mathbb{Z}^2$ such that:*

(1) $1-n \leq y_i \leq 1$ *and* $\sum\limits_{i=1}^{n+1} y_i = 1$

(2) $1 \leq x_i \leq m-1$ *and* $\sum\limits_{i=1}^{n+1} x_i = m$.

(x_i,y_i) is called the steps of P for any $i \in [n+1]$. Since P can be viewed as a path from the origin to $(m,1)$ in the plane $\mathbb{Z}\times\mathbb{Z}$ and has $n+1$ steps, we say that P is of order $n+1$ and length m.

2.1 The non-positive length of an (n,m)-lattice paths

Given an (n,m)-lattice path $P=(x_1,y_1)(x_2,y_2)\dots(x_{n+1},y_{n+1})$, we let $NP(P)=\{i \mid \sum\limits_{j=1}^{i} y_j \leq 0\}$ and $NPL(P)=\sum\limits_{i\in NP(P)} x_i$. Clearly, $0 \leq NPL(P) \leq m-x_{n+1} \leq m-1$ since $n+1 \neq NP(P)$. We say that $NPL(P)$ is the *non-positive length* of the (n,m)-lattice path P. Moreover, we define a linear order $<_P$ on the set $[n+1]$ by the following rules:

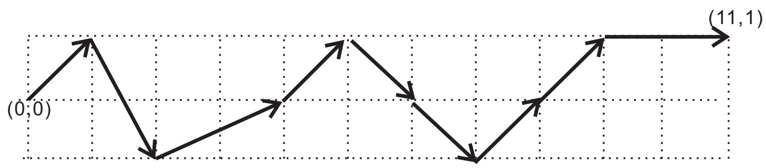
for any $i,j\in[n+1]$, $i <_P j$ if either (1) $\sum\limits_{k=1}^{i} y_k < \sum\limits_{k=1}^{j} y_k$ or (2) $\sum\limits_{k=1}^{i} y_k = \sum\limits_{k=1}^{j} y_k$ and $i>j$.

The sequence formed by writing $[n+1]$ in the increasing order with respect to $<_P$ is denoted by $\pi_P=(\pi_P(1),\pi_P(2),\dots,\pi_P(n+1))$.

Example 2.2 *Let $n=8$ and $m=11$. We draw an $(8,11)$-lattice path*

$$P=(1,1)(1,-2)(2,1)(1,1)(1,-1)(1,-1)(1,1)(1,1)(2,0)$$

as follows.



Then $NP(P) = \{2, 3, 5, 6, 7\}$*,* $NPL(P) = 6$ *and* $\pi_P = (6, 2, 7, 5, 3, 9, 8, 4, 1)$*.*

We use $\mathcal{L}_{n,m,r}$ to denote the set of all the (n, m)-lattice paths P such that $NPL(P) = r$. In particularly, we use $\tilde{\mathcal{L}}_{n,m,0}$ to denote the set of all the lattice paths $P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})$ in the set $\mathcal{L}_{n,m,0}$ such that $x_{n+1} = 1$. Clearly, $\tilde{\mathcal{L}}_{n,m,0} \subset \mathcal{L}_{n,m,0}$.

Lemma 2.3

(1) The number of the (n, m)*-lattice paths* P *such that* $NPL(P) = 0$ *is equal to* $\binom{m-1}{n}c_n$*;*
(2) The number of the (n, m)*-lattice paths* $P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})$ *such that* $NPL(P) = 0$ *and* $x_{n+1} = 1$ *is equal to* $\binom{m-2}{n-1}c_n$*.*

Proof. (1) It is well known that the number of the solutions of the equation $\sum\limits_{i=1}^{n+1} y_i = 1$ such that $1 - n \leq y_i \leq 1$ and $NP(P) = \emptyset$ is c_n and the number of the solutions of the equation $\sum\limits_{i=1}^{n+1} x_i = m$ in positive integers is $\binom{m-1}{n}$. Hence, The number of the (n, m)-lattice paths P such that $NPL(P) = 0$ is equal to $\binom{m-1}{n}c_n$.

(2) Note that the number of the solutions of the equation $\sum\limits_{i=1}^{n} x_i = m - 1$ in positive integers is $\binom{m-2}{n-1}$. We immediately obtain that the number of the (n, m)-lattice paths P such that $NPL(P) = 0$ and x_{n+1} is equal to $\binom{m-2}{n-1}c_n$. ∎

Lemma 2.4 *There is a bijection* Φ *from* $\mathcal{L}_{n,m,r}$ *to* $\mathcal{L}_{n,m,r+1}$ *for any* $1 \leq r \leq m - 2$*.*

Proof. Let $P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1}) \in \mathcal{L}_{n,m,r}$. Consider the sequence π_P. Suppose $\pi_P(k) = n + 1$ for some k. Since $r \geq 1$, we have $k \geq 2$. We discuss the following two cases.

Case I. $k \leq n$

If $x_{n+1} = 1$, then let $i = \pi_P(k + 1)$ and

$$\Phi(P) = (x_{i+1}, y_{i+1}) \ldots (x_{n+1}, y_{n+1})(x_1, y_1) \ldots (x_i, y_i).$$

If $x_{n+1} \geq 2$, then let $i = \pi_P(k - 1)$ and

$$\Phi(P) = (x_1, y_1) \ldots (x_i + 1, y_i) \ldots (x_{n+1} - 1, y_{n+1}).$$

Case II. $k = n + 1$

Note that $x_{n+1} \geq 2$ since $r \leq m-2$. We let $i = \pi_P(n)$ and

$$\Phi(P) = (x_1, y_1) \dots (x_i + 1, y_i) \dots (x_{n+1} - 1, y_{n+1}).$$

It is easy to see that $\Phi(P) \in \mathcal{L}_{n,m,r+1}$ for Cases I and II.

For proving that Φ is a bijection, we describe the inverse of Φ as follows.

Let $P = (x_1, y_1)(x_2, y_2) \dots (x_{n+1}, y_{n+1}) \in \mathcal{L}_{n,m,r+1}$, where $1 \leq r \leq m-2$. Suppose $\pi_P(k) = n+1$ for some k. Let $i = \pi_P(k-1)$. If $x_i = 1$, then let

$$\Phi^{-1}(P) = (x_{i+1}, y_{i+1}) \dots (x_{n+1}, y_{n+1})(x_1, y_1) \dots (x_i, y_i);$$

otherwise, let

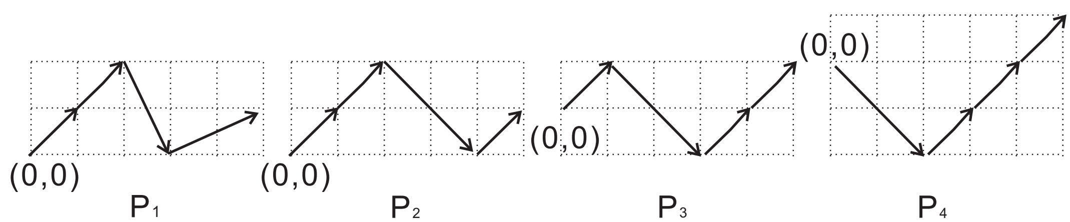
$$\Phi^{-1}(P) = (x_1, y_1) \dots (x_i - 1, y_i) \dots (x_{n+1} + 1, y_{n+1}).$$

This complete the proof. ∎

Example 2.5 *Let $n = 3$ and $m = 5$. We draw $(3,5)$-lattice paths*

$$P_1 = (1,1)(1,1)(1,-2)(2,1) \quad P_2 = (1,1)(1,1)(2,-2)(1,1)$$
$$P_3 = (1,1)(2,-2)(1,1)(1,1) \quad P_4 = (2,-2)(1,1)(1,1)(1,1)$$

as follows.



We have $\Phi(P_i) = P_{i+1}$ and $NPL(P_i) = i$.

Lemma 2.6 *There is a bijection from $\tilde{\mathcal{L}}_{n,m,0}$ to $\mathcal{L}_{n,m,1}$.*

Proof. Let $P = (x_1, y_1)(x_2, y_2) \dots (x_{n+1}, y_{n+1}) \in \tilde{\mathcal{L}}_{n,m,0}$. Consider the sequence π_P. Note that $\pi_P(1) = n+1$ for any $P \in \mathcal{L}_{n,m,0}$. So, let $i = \pi_P(2)$. Let the mapping Φ be defined as that in Lemma 2.4, i.e., $\Phi(P) = (x_{i+1}, y_{i+1}) \dots (x_{n+1}, y_{n+1})(x_1, y_1) \dots (x_i, y_i)$. Then $\Phi(P) \in \mathcal{L}_{n,m,1}$. Conversely, for any $P = (x_1, y_1)(x_2, y_2) \dots (x_{n+1}, y_{n+1}) \in \mathcal{L}_{n,m,1}$, we have $\pi_P(2) = n+1$. Suppose $\pi_P(1) = i$, then $x_i = 1$. This tells us that Φ is a bijection from $\tilde{\mathcal{L}}_{n,m,0}$ to $\mathcal{L}_{n,m,1}$. ∎

Theorem 2.7 *For any $1 \leq r \leq m-1$, the number of the (n,m)-lattice paths P such that $NPL(P) = r$ is equal to the number of the (n,m)-lattice paths $P = (x_1, y_1)(x_2, y_2) \dots (x_{n+1}, y_{n+1})$ such that $NPL(P) = 0$ and $x_{n+1} = 1$ and independent on r.*

Proof. Combining Lemmas 2.4 and 2.6, we immediately obtain the results as desired. ∎

2.2 The rightmost minimum length of an (n,m)-lattice paths

Given a (n,m)-lattice path $P=(x_1,y_1)(x_2,y_2)\dots(x_{n+1},y_{n+1})$, we let $a_0=0$, $b_0=0$, $a_i=\sum_{j=1}^{i} y_j$ and $b_i=\sum_{j=1}^{i} x_j$ for $i\geq 1$. Then the (n,m)-lattice path P can be viewed as a sequence of the points in the plane $\mathbb{Z}\times\mathbb{Z}$

$$(b_0,a_0),(b_1,a_1),\dots,(b_{n+1},a_{n+1}).$$

A *minimum point* of the path P is a point (b_i,a_i) such that $a_i\leq a_j$ for all $j\neq i$. A *rightmost minimum point* is a minimum point (b_i,a_i) such that the point is the rightmost one among all the minimum points. If (b_i,a_i) is the minimum point of the path P, we call b_i the *rightmost minimum length* of the (n,m)-lattice paths P, denoted by $RML(P)$.

Example 2.8 *We consider the path P in Example 2.2. The point $(7,-1)$ is the rightmost minimum point and $RML(P)=7$.*

We use $\mathcal{M}_{n,m,r}$ to denote the set of all the (n,m)-lattice paths P such that $RML(P)=r$.

Lemma 2.9 *There is a bijection Ψ from $\mathcal{M}_{n,m,r}$ to $\mathcal{M}_{n,m,r+1}$ for any $1\leq r\leq m-2$.*

Proof. Let $P=(x_1,y_1)(x_2,y_2)\dots(x_{n+1},y_{n+1})\in\mathcal{M}_{n,m,r}$. If $x_{n+1}=1$, we let

$$\Psi(P)=(x_{n+1},y_{n+1})(x_1,y_1)\dots(x_n,y_n);$$

otherwise let

$$\Psi(P)=(x_1+1,y_1)(x_2,y_2)\dots(x_n,y_n)(x_{n+1}-1,y_{n+1}).$$

It is easy to see that $\Psi(P)\in\mathcal{M}_{n,m,r+1}$.

For proving that Φ is a bijection, we describe the inverse of Φ as follows.

If $x_1=1$, we let

$$\Psi(P)=(x_2,y_2)(x_3,y_3)\dots(x_{n+1},y_{n+1})(x_1,y_1);$$

otherwise let

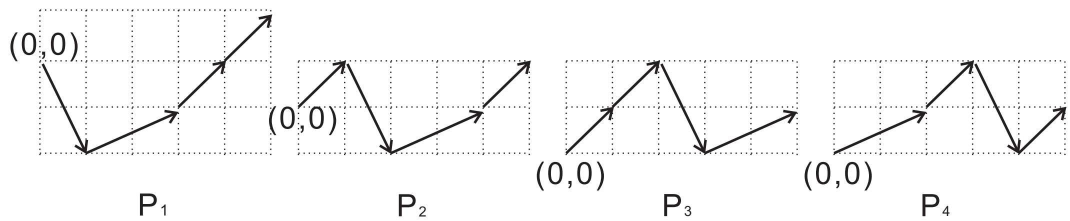
$$\Psi(P)=(x_1-1,y_1)(x_2,y_2)\dots(x_n,y_n)(x_{n+1}+1,y_{n+1}).$$

This complete the proof. ■

Example 2.10 *Let $n=3$ and $m=5$. We draw $(3,5)$-lattice paths*

$$P_1=(1,-2)(2,1)(1,1)(1,1)\quad P_2=(1,1)(1,-2)(2,1)(1,1)$$
$$P_3=(1,1)(1,1)(1,-2)(2,1)\quad P_4=(2,1)(1,1)(1,-2)(1,1)$$

as follows.



We have $\Psi(P_i) = P_{i+1}$ *and* $RML(P_i) = i$.

Note that $NPL(P) = 0$ if and only if $RML(P) = 0$ for any (n,m)-lattice path. Recall that $\tilde{\mathcal{L}}_{n,m,0}$ is the set of all the lattice paths $P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})$ in the set $\mathcal{L}_{n,m,0}$ such that $x_{n+1} = 1$. Hence, also $\tilde{\mathcal{L}}_{n,m,0}$ is the set of all the lattice paths $P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})$ in the set $\mathcal{M}_{n,m,0}$ such that $x_{n+1} = 1$.

Lemma 2.11 *There is a bijection from* $\tilde{\mathcal{L}}_{n,m,0}$ *to* $\mathcal{M}_{n,m,1}$.

Proof. Let $P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1}) \in \tilde{\mathcal{L}}_{n,m,0}$. Then $x_{n+1} = 1$ and $y_{n+1} \leq 0$. We let

$$\Psi(P) = (x_{n+1}, y_{n+1})(x_1, y_1) \ldots (x_n, y_n).$$

Clearly, $\Psi(P) \in \mathcal{M}_{n,m,1}$.

Conversely, let $P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1}) \in \tilde{\mathcal{L}}_{n,m,1}$. Then $x_1 = 1$ and $y_1 \leq 0$. We let

$$\Psi(P) = (x_2, y_2)(x_3, y_3) \ldots (x_{n+1}, y_{n+1})(x_1, y_1).$$

This complete the proof. ∎

Theorem 2.12 *For any* $1 \leq r \leq m-1$, *the number of the* (n,m)*-lattice paths* P *such that* $RML(P) = r$ *is equal to the number of the* (n,m)*-lattice paths* $P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})$ *such that* $RML(P) = 0$ *and* $x_{n+1} = 1$ *and independent on* r.

Proof. Combining Lemmas 2.9 and 2.11, we immediately obtain the results as desired. ∎

3 The pointed (n,m)-lattice path

In this section, we will consider the pointed (n,m)-lattice paths. We will define two parameters for an pointed (n,m)-lattice path: the pointed non-positive length and the pointed rightmost minimum length. We will give generalizations of the results in [5]. We will prove the Chung-Feller theorems of the pointed (n,m)-lattice path on these two parameters. First, we give the definition of the pointed (n,m)-lattice paths as follows.

Definition 3.1 *A pointed (n,m)-lattice paths $\dot{P}$ is a pair $[P;j]$ such that:*

(1) $P=(x_1,y_1)(x_2,y_2)\ldots(x_{n+1},y_{n+1})$ is an (n,m)-lattice paths;

(2) $0\le j\le x_{n+1}-1$.

We call the point $(m-j,0)$ the root of P. We use $\mathscr{L}_{n,m}$ to denote the set of the pointed (n,m)-lattice paths.

Lemma 3.2 *The number of the pointed (n,m)-lattice paths is $\binom{2n}{n}\binom{m}{n+1}$.*

Proof. Note that the number of the solutions of the equation $\sum\limits_{i=1}^{n+1} y_i=1$ such that $1-n\le y_i\le 1$ is $\binom{2n}{n}$. On the other hand, we let $z_i=x_i$ for all $i\in[n]$, $z_{n+1}=x_{n+1}-j$ and $z_{n+2}=j$. Since $\sum\limits_{i=1}^{n+1} x_i=m$, $x_i\ge 1$ and $0\le j\le x_{n+1}-1$, we have $\sum\limits_{i=1}^{n+2} z_i=m$, $z_i\ge 1$ for all $i\in[n+1]$ and $z_{n+2}\ge 0$. It is easy to see that the number of the solutions of the equation $\sum\limits_{i=1}^{n+2} z_i=m$ such that $z_i\ge 1$ for all $i\in[n+1]$ and $z_{n+2}\ge 0$ is $\binom{m}{n+1}$. Hence, the number of the pointed (n,m)-lattice paths is $\binom{2n}{n}\binom{m}{n+1}$. ∎

3.1 The pointed non-positive length of an pointed (n,m)-lattice paths

Given a pointed (n,m)-lattice path $\dot{P}=[P;j]$, where $P=(x_1,y_1)(x_2,y_2)\ldots(x_{n+1},y_{n+1})$ and $0\le j\le x_{n+1}-1$, we let $PNPL(\dot{P})=NPL(P)+j$. Clearly, $0\le PNPL(\dot{P})\le m-1$. We say that $PNPL(\dot{P})$ is the *pointed non-positive length* of the path $\dot{P}$.

By Lemma 2.3 (1), we have the following lemma.

Lemma 3.3 *The number of the pointed (n,m)-lattice paths with pointed non-positive length 0 is $\binom{m-1}{n}c_n$.*

Given an (n,m)-lattice path $P=(x_1,y_1)(x_2,y_2)\ldots(x_{n+1},y_{n+1})$, we let

$$P_i=(x_{i+1},y_{i+1})\ldots(x_{n+1},y_{n+1})(x_1,y_1)\ldots(x_i,y_i).$$

P_i is call the *ith cyclic permutation* of P. Furthermore, setting the point $(m-j,0)$ to be the root of P_i, where $0\le j\le x_i-1$, we get a pointed (n,m)-lattice paths $[P_i;j]$, denoted by $\dot{P}(i;j)$. Finally, we define a set $\mathcal{PL}(P)$ as follows:

$$\mathcal{PL}(P)=\{\dot{P}(i;j)\mid i\in[n+1] \text{ and } 0\le j\le x_i-1\}.$$

Clearly, we have the following lemma.

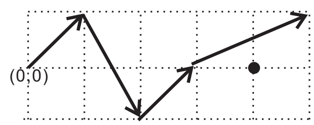
Lemma 3.4 *$|\mathcal{PL}(P)|=m$.*

Recall that $<_P$ is the linear order on the set $[n+1]$. We define a linear order $\prec_P$ on the set $\mathcal{PL}(P)$ by the following rules:

for any $\dot{P}(i_1; j_1), \dot{P}(i_2; j_2) \in \mathcal{PL}(P)$, $\dot{P}(i_1; j_1) \prec_P \dot{P}(i_2; j_2)$ if either (1) $i_1 <_P i_2$ or (2) $i_1 = i_2$ and $j_1 < j_2$.

The sequence, which is formed by the elements in the set $\mathcal{PL}(P)$ in the increasing order with respect to $\prec_P$, reduce a bijection from the sets $[m]$ to $\mathcal{PL}(P)$, denoted by $\Theta = \Theta_P$.

Example 3.5 *Let $n = 3$ and $m = 5$. Let $P = (1,1)(1,-2)(1,1)(2,1)$. We draw the pointed $(3,5)$-lattice path $\dot{P} = [P; 1]$ as follows.*



where the root is the point $(4,0)$ denoted by $\bullet$. Then $PNPL(\dot{P}) = 3$. We write the bijection Θ_P as the following 2×5 matrix.

$$\Theta_P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \dot{P}(2;0) & \dot{P}(3;0) & \dot{P}(4;0) & \dot{P}(4;1) & \dot{P}(1;0) \end{pmatrix}$$

Theorem 3.6 *Let P be an (n,m)-lattice path, $\mathcal{PL}(P)$ and Θ_P defined as above. Then*

$$PNPL(\Theta(r)) = r - 1$$

for any $r \in [m]$.

Proof. Note that $0 \leq PNPL(\Theta(r)) \leq m - 1$ for any $r \in [m]$. It is sufficient to prove that $PNPL(\Theta(r+1)) = PNPL(\Theta(r)) + 1$ for any $r \in [m-2]$. Suppose

$$P = (x_1, y_1)(x_2, y_2) \ldots (x_{n+1}, y_{n+1})$$

and $\Theta(r) = \dot{P}(s; t) \in \mathcal{PL}(P)$. Let π_P be the sequence formed by writing $[n+1]$ in the increasing order with respect to $<_P$ and $\pi_P^{-1}(s) = k$. Then $PNPL(\Theta(r)) = \sum\limits_{j=1}^{k-1} x_{\pi_P(j)} + t$. Now, suppose $\Theta(r+1) = \dot{P}(\tilde{s}; \tilde{t})$. We discuss the following two cases:

Case I. $s = \tilde{s}$

Then $\tilde{t} = t + 1$. This implies $PNPL(\Theta(r+1)) = PNPL(\Theta(r)) + 1$.

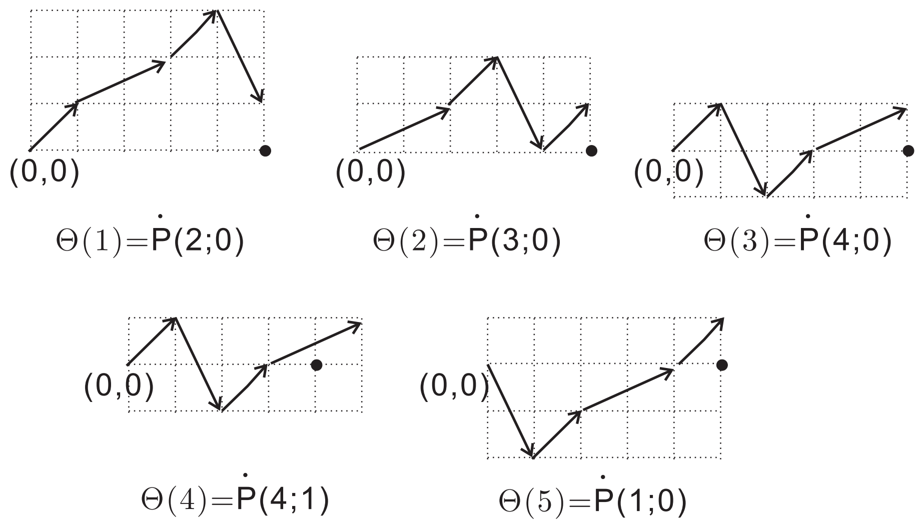
Case II. $s <_P \tilde{s}$

Then $\pi_P(k+1) = \tilde{s}$, $t = x_s - 1$ and $\tilde{t} = 0$. Thus,

$$PNPL(\Theta(r+1)) = \sum_{j=1}^{k} x_{\pi_P(j)} = \sum_{j=1}^{k-1} x_{\pi_P(j)} + x_s = PNPL(\Theta(r)) + 1.$$

This complete the proof. ∎

Example 3.7 *We consider the path P in Example 3.5. We draw the pointed lattice path $\Theta(r)$ as follows:*



Remark 3.8 *Let $\dot{P} = [P; j]$ be a pointed (n, m)-lattice path, where $P = (x_1, y_1) \dots (x_{n+1}, y_{n+1})$ and $0 \le j \le x_{n+1} - 1$. Setting $m = n+1$, we have $x_i = 1$ for all i and $j = 0$. Let $Y = (y_1, \dots, y_{n+1})$. Then $E(Y) = PNPL(\dot{P})$. This tells us that Lemma 1.1 can be viewed as a corollary of Theorem 3.6.*

We use $\mathscr{L}_{n,m,r}$ to denote the set of the pointed (n, m)-lattice paths with pointed non-positive length r. Clearly, $\mathscr{L}_{n,m} = \bigcup_{r=0}^{m-1} \mathscr{L}_{n,m,r}$. Let $l_{n,m,r} = |\mathscr{L}_{n,m,r}|$.

Corollary 3.9 *For any $0 \le r \le m-1$, the number of the pointed (n, m)-lattice paths with pointed non-positive length r is equal to the number of the pointed (n, m)-lattice paths with pointed non-positive length 0 and independent on r, i.e., $l_{n,m,r} = \frac{1}{m}\binom{2n}{n}\binom{m}{n+1}$.*

Proof. First, we define an equivalent relation on the set $\mathscr{L}_{n,m}$. Let $\dot{P} = [P; i]$ and $\dot{Q} = [Q; j]$ be two pointed (n, m)-lattice paths. Suppose $P = (x_1, y_1) \dots (x_{n+1}, y_{n+1})$. Recall P_k denote the kth cyclic permutation of P, i.e., $P_k = (x_{k+1}, y_{k+1}) \dots (x_{n+1}, y_{n+1})(x_1, y_1) \dots (x_k, y_k)$. We say $\dot{Q}$ and $\dot{P}$ is equivalent, denoted by $\dot{Q} \sim \dot{P}$, if $Q = P_k$ for some $k \in [n+1]$. Hence, given a pointed lattice path $\dot{P} \in \mathscr{L}_{n,m}$, we define a set $EQ(\dot{P})$ as $EQ(\dot{P}) = \{\dot{Q} \in \mathscr{L}_{n,m} \mid \dot{Q} \sim \dot{P}\}$. We say that the set $EQ(\dot{P})$ is an equivalent class of the set $\mathscr{L}_{n,m}$. Clearly, $|EQ(\dot{P})| = m$. Now, we may suppose that the set $\mathscr{L}_{n,m}$ has t equivalent class. Then $t = \frac{1}{m}\binom{2n}{n}\binom{m}{n+1}$. For any $0 \le r \le m-1$, from Theorem 3.6, every equivalent class contains exactly one element with pointed non-positive length r. Hence, $l_{n,m,r} = t = \frac{1}{m}\binom{2n}{n}\binom{m}{n+1}$. ∎

3.2 The pointed rightmost minimum length of an pointed (n,m)-lattice paths

Let $\dot{P}=[P;j]$ be a pointed (n,m)-lattice path, where $P=(x_1,y_1)(x_2,y_2)\dots(x_{n+1},y_{n+1})$ is a (n,m)-lattice path and $0\leq j\leq x_{n+1}-1$. Recall that $RML(P)$ is the rightmost minimum length of P. We let $PRML(\dot{P})=RML(P)+j$ and call $PRML(\dot{P})$ the *pointed rightmost minimum length* of $\dot{P}$.

Note that $PNPL(P)=0$ if and only if $PRML(P)=0$ for any pointed (n,m)-lattice path. We immediately obtain the following lemma.

Lemma 3.10 *The number of the pointed (n,m)-lattice paths with pointed rightmost minimum length 0 is $\binom{m-1}{n}c_n$.*

First, given a (n,m)-lattice path P, we recall that π_P is the sequence formed by writing $[n+1]$ in the increasing order with respect to $<_P$. Suppose $\pi_P(1)=i$. Let $\sigma_P=(\sigma_P(1),\sigma_P(2),\dots,\sigma_P(n+1))=(i,i-1,\dots,1,n+1,n,\dots,i+1)$.

Using σ_P, we define a new linear order $\prec_P^*$ on the set $\mathcal{PL}(P)=\{\dot{P}(i;j)\mid i\in[n+1]\text{ and }0\leq j\leq x_i-1\}$ by the following rules:

for any $\dot{P}(i_1;j_1),\dot{P}(i_2;j_2)\in\mathcal{PL}(P)$, $\dot{P}(i_1;j_1)\prec_P^*\dot{P}(i_2;j_2)$ if either (1) $\sigma_P^{-1}(i_1)<\sigma_P^{-1}(i_2)$ or (2) $i_1=i_2$ and $j_1<j_2$.

The sequence, which is formed by the elements in the set $\mathcal{PL}(P)$ in the increasing order with respect to $\prec_P^*$, reduce a bijection from the sets $[m]$ to $\mathcal{PL}(P)$, denoted by $\Gamma=\Gamma_P$.

Example 3.11 *Consider the path P and the pointed path $\dot{P}$ in Example 3.5. we have $PRML(\dot{P})=3$. It is easy to see $\sigma_P=(2,1,4,3)$. We write the bijection Γ_P as the following 2×5 matrix.*

$$\Gamma_P=\begin{pmatrix} 1 & 2 & 3 & 4 & 5\\ \dot{P}(2;0) & \dot{P}(1;0) & \dot{P}(4;0) & \dot{P}(4;1) & \dot{P}(3;0)\end{pmatrix}$$

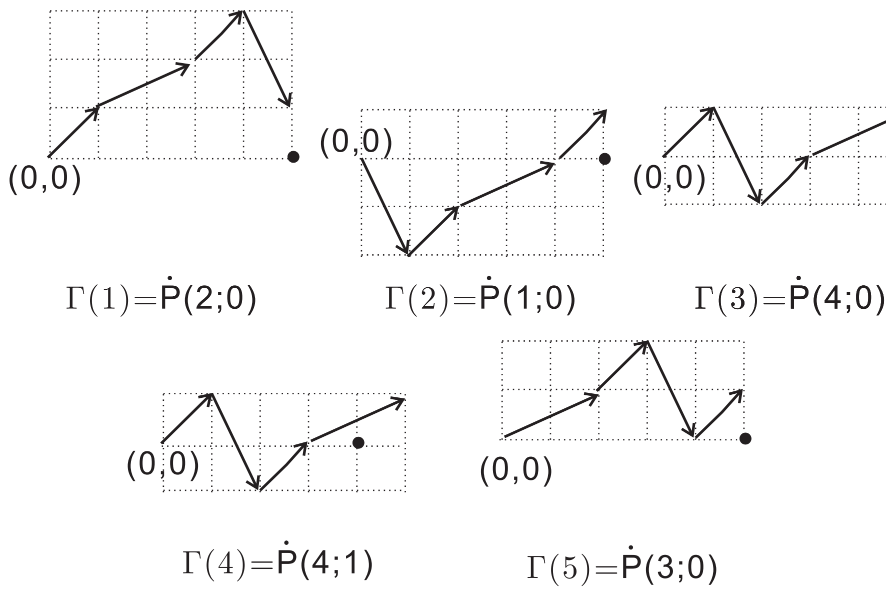
Theorem 3.12 *Let P be an (n,m)-lattice path and Γ defined as above. Then*

$$PRML(\Gamma(r))=r-1$$

for any $r\in[m]$.

Proof. It is sufficient to prove that $PRML(\Gamma(r+1))=PRML(\Gamma(r))+1$. Suppose $\Gamma(r)=\dot{P}(i_1;j_1)$ and $\Gamma(r+1)=\dot{P}(i_2;j_2)$. If $i_1=i_2$, then $j_1+1=j_2$. Clearly, $PRML(\Gamma(r+1))=PRML(\Gamma(r))+1$. We consider the case with $\sigma_P^{-1}(i_1)<\sigma_P^{-1}(i_2)$. Let $k=\sigma_P^{-1}(i_1)$. Then $\sigma_P^{-1}(i_2)=k+1$, $j_1=x_{i_1}-1$ and $j_2=0$. We have $PRML(\dot{P}(i_2;j_2))=\sum\limits_{j=1}^{k}x_{\sigma_P(j)}=\sum\limits_{j=1}^{k-1}x_{\sigma_P(j)}+x_{i_1}=PRML(\dot{P}(i_1;j_1))+1$. ∎

Example 3.13 *We consider the path P in Example 3.5. We draw the pointed lattice path $\Gamma(r)$ as follows:*



We use $\mathscr{M}_{n,m,r}$ to denote the set of the pointed (n,m)-lattice paths with pointed rightmost minimum length r. Clearly, $\mathscr{L}_{n,m} = \bigcup_{r=0}^{m-1} \mathscr{M}_{n,m,r}$. Let $d_{n,m,r} = |\mathscr{M}_{n,m,r}|$.

Corollary 3.14 *For any $0 \le r \le m-1$, the number of the pointed (n,m)-lattice paths with pointed rightmost minimum length r is equal to the number of the pointed (n,m)-lattice paths with pointed rightmost minimum length 0 and independent on r, i.e., $d_{n,m,r} = \frac{1}{m}\binom{2n}{n}\binom{m}{n+1}$.*

Proof. Similar to the proof of Corollary 3.9, we can obtain the results as desired. ∎

4 The application of the main theorem

In fact, by Theorems 3.6 and 3.12, we may find the Chung-Feller theorems of many different (n,m)-lattice paths on the parameter: the pointed non-positive length and the pointed rightmost minimum length. For example, we let A and B be two finite subsets of the set $\mathbb{P}$. Let $\mathcal{S} = \mathcal{S}_A \cup \mathcal{S}_B \cup \{(1,1)\}$, where $\mathcal{S}_A = \{(2i-1,-1) \mid i \in A\}$ and $\mathcal{S}_B = \{(2i,0) \mid i \in B\}$. In [6], we have proved the following corollary by the generating function methods. Using Theorems 3.6 and 3.12, we can reobtain the corollary.

Corollary 4.1 *Let $\mathscr{P}_{n,m}$ be the set of the pointed lattice paths in the plane $\mathbb{Z} \times \mathbb{Z}$ which (1) only use steps in the set $\mathcal{S}$; (2) have $n+1$ steps; (3) go from the origin to the point (m,1). Then in $\mathscr{P}_{n,m}$,*

(1) the number of the pointed lattice paths with pointed non-positive length r is equal to the number of the pointed lattice paths with pointed non-positive length 0 and independent on r;
(2) the number of the pointed lattice paths with pointed rightmost minimum length r is equal to the number of the pointed lattice paths with pointed rightmost minimum length 0 and independent on r.

Proof. (1) It is easy to see that a pointed lattice path P in $\mathscr{P}_{n,m}$ can be view as a pointed (n, m)-lattice path $(x_1, y_1) \dots (x_{n+1}, y_{n+1})$ such that $(x_i, y_i) \in \mathcal{S}$ for all $i \in [n+1]$. By Theorem 3.6, using a similar method as Corollary 3.9, we get the results as desired.
(2) The proof is omitted. ∎

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References

[1] Y.M. Chen, The Chung-Feller theorem revisited, *Discrete Math. 308 (2008) 1328-1329*

[2] K.L. Chung, W. Feller, On fluctuations in-coin tossing, *Proc. Natl. Acad. Sci. USA 35 (1949) 605-608*

[3] S. P. Eu, T.S.Fu, Y.N. Yeh, Taylor expansions for Catalan and Motzkin numbers, *Adv. Appl. Math. 29 (2002) 345-357*

[4] S. P. Eu, T.S.Fu, Y.N. Yeh, Refined Chung-Feller theorems for lattice paths, *J. Combin. Theory Ser. A 112 (2005) 143-162*

[5] Sri Gopal Mohanty, Lattice path counting and applications, *New York : Academic Press, 1979*

[6] Jun Ma, Yeong-nan Yeh, Generalizations of The Chung-Feller Theorem, submitted.

[7] P.A. MacMahon, Memoir on the theory of the partitions of numbers, *Philos. Trans. Roy. Soc. London, Ser. A, 209 (1909), 153-175*; Also G.E. Andrews (Ed.), *Collected Works, vol. 1, MIT Press, Cambridge, MA, 1978, 1292-1314.*

[8] T.V.Narayana, Cyclic permutation of lattice paths and the Chung-Feller theorem, *Skand. Aktuarietidskr. (1967) 23-30*

[9] L. Shapiro, Some open questions about random walks, involutions, limiting distributions, and generating functions, *Advances in Applied Math. 27 (2001), 585-596.*